7 Representability of the Picard functor II / 17.05.22 / Felix Lotter

In this talk, k will always be an algebraically closed field and X a smooth projective curve over k.
Recall that we defined the Picard functor $\mathrm{Pic}_{X/k}$ by the exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \xrightarrow{p_T^*} \mathrm{Pic}(X_T) \longrightarrow \mathrm{Pic}_{X/k}(T) \longrightarrow 0$$

for every scheme T over k, where p is the structure morphism $X \to k$. Pick a k-rational point of X, then this gives a map $\sigma : k \to X$ which is a section of p, so in this case $\mathrm{Pic}_{X/S}(T) \cong \ker(\mathrm{Pic}(X_T) \xrightarrow{\sigma_T^*} \mathrm{Pic}(T)) =: \mathrm{Pic}_{X/S,\sigma}(T)$.

Notation 7.1. If X, T are schemes over k, then we denote by X_T the base change $X \times_k T$. If K is a field, then we will simply write K for both $\operatorname{Spec} K$ and K.

7.1 Representing the Picard functor

We want to use the following representability criterion (which is similar to the one we know from Talk 3):

Lemma 7.2 (Stacks [0B9Q])**.** *Let k be a field. Let $G : (Sch/k)^{opp} \to Groups$ be a functor. Assume that*

1. *G satisfies the sheaf property for the Zariski topology,*
2. *there exists a subfunctor $F \subset G$ such that*
   (a) *F is representable and open*
   (b) *for every field extension K of k and $g \in G(K)$ there exists a $g' \in G(k)$ such that $g'g \in F(K)$.*

Then G is representable by a group scheme over k.

In Talk 6, we constructed the subfunctor F of $\mathrm{Pic}_{X/S,\sigma}(T)$ and proved that it satisfies $2(a)$. It remains to show 1 and $2(b)$.

Proposition 7.3. $\mathrm{Pic}_{X/k,\sigma}$ *has the sheaf property for the Zariski topology.*

Proof. Let $T = \bigcup_{i\in I} T_i$ be an open cover, define $T_{ij} := T_i \cap T_j$ and let $\mathcal{L}_i$ be elements of $\mathrm{Pic}_{X/k,\sigma}(T_i)$, that is, line bundles on X_{T_i} with $\sigma_{T_i}^* \mathcal{L}_i \cong \mathcal{O}_{T_i}$. Fix such an isomorphism $\alpha_i : \mathcal{O}_{T_i} \to \sigma_{T_i}^* \mathcal{L}_i$ for every i. Assume $\mathcal{L}_i$ and $\mathcal{L}_j$ map to the same elements of $\mathrm{Pic}_{X/k,\sigma}(T_{ij})$ for all i and j. That is, we can choose isomorphisms

$$\phi_{ij} : \mathcal{L}_i|_{X_{T_{ij}}} \to \mathcal{L}_j|_{X_{T_{ij}}}$$

for all i, j.
Now consider the automorphisms

$$\alpha_j|_{T_{ij}}^{-1} \circ \sigma_{T_{ij}}^* \phi_{ij} \circ \alpha_i|_{T_{ij}}$$

of $\mathcal{O}_{T_{ij}}$. They are given by multiplication with some unit $u_{ij} \in \Gamma(T_{ij}, \mathcal{O}_{T_{ij}})^\times$. By scaling ϕ_{ij} with u_{ij}^{-1}, we may assume w.l.o.g. that all u_{ij} are equal to 1, i.e. that $\sigma_{T_{ij}}^* \phi_{ij} = \alpha_j|_{T_{ij}} \alpha_i|_{T_{ij}}^{-1}$. Now for $i, j, k \in I$ define $T_{ijk} := T_{ij} \cap T_k$. Then

$$\sigma_{T_{ijk}}^* (\phi_{ki}|_{X_{T_{ijk}}} \circ \phi_{jk}|_{X_{T_{ijk}}} \circ \phi_{ij}|_{X_{T_{ijk}}}) = \alpha_i|_{T_{ijk}} \alpha_i|_{T_{ijk}}^{-1} = id$$

Now use the following important fact: $\Gamma(X_T, \mathcal{O}_{X_T}) \cong \Gamma(T, \mathcal{O}_T)$ via f_T^* and σ_T^* (This is follows from flat base change). Since automorphisms of $\sigma_{T_{ijk}}^* \mathcal{L}_i|_{X_{T_{ijk}}}$ correspond to elements of $\Gamma(X_{T_{ijk}}, \mathcal{O}_{X_{T_{ijk}}})^\times$, it follows that $\phi_{ki}|_{X_{T_{ijk}}} \circ \phi_{jk}|_{X_{T_{ijk}}} \circ \phi_{ij}|_{X_{T_{ijk}}} = id$; by the same argument we see that $\phi_{ki}|_{X_{T_{ijk}}} = \phi_{ik}|_{X_{T_{ijk}}}^{-1}$ and thus, $\phi_{ik}|_{X_{T_{ijk}}} = \phi_{jk}|_{X_{T_{ijk}}} \circ \phi_{ij}|_{X_{T_{ijk}}}$. So the $\mathcal{L}_i$ glue to a line bundle $\mathcal{L}$ on X_T with $\sigma_{T_{ijk}}^* \mathcal{L} \cong \mathcal{O}_T$. $\square$

Recall our definition of the subfunctor F: For a scheme $T = \mathrm{Spec}(R)$ over k, $F(T) \subseteq \mathrm{Pic}_{X/k}(T)$ consists of those line bundles $\mathcal{L} \in \mathrm{Pic}_{X/k}(T)$ such that

$$H^i(X_T, \mathcal{L}) = \begin{cases} 0 & i > 0 \\ \text{invertible } R\text{-module} & i = 0 \end{cases}$$

.

So we need to check the following for $2(c)$:

Lemma 7.4. *Let K/k be a field extension and let $\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(K)$. Then there exists a line bundle $\mathcal{L}_0 \in \mathrm{Pic}_{X/k,\sigma}(k)$ such that $\dim_K H^0(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 1$ and $\dim_K H^1(X_K, \mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}) = 0$.*

Proof. We pick an ample line bundle $\mathcal{L}_0$ on X and replace $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}_0|_{X_K}^{\otimes n}$. Note that $\mathcal{L}_0|_{X_K}$ is still an ample line bundle, as $X_K \to X$ is affine. We choose $n >> 0$ such that $H^0(X_K, \mathcal{L}) \neq 0$ and $H^i(X_K, \mathcal{L}) = 0$ for all $i > 0$. That this is possible will probably be proven in the lecture (it is Theorem 15.2 in Scholze's Algebraic Geometry II notes).
Now the idea is to inductively reduce the dimension of $H^0(X_K, \mathcal{L})$. Assume $t := \dim_K H^0(X_K, \mathcal{L}) > 1$.
We know that we have infinitely many k-rational points on X (as k is algebraically closed). Moreover, if x is a k-rational point, then x_K is a K-rational point. Thus, the points x_K form a Zariski-dense subset of X_K (the topology on X_K is cofinite).
Now let $s \in H^0(X_K, \mathcal{L})$ be nonzero. Then there is some k-rational point x such that s does not vanish in x_K. Let $\mathcal{I}$ be the ideal sheaf of $i : x_K \to X_K$, i.e. we have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{X_K} \longrightarrow i_*\mathcal{O}_{x_K} \longrightarrow 0$$

Tensoring this with $\mathcal{L}$ we get

$$0 \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow i_*\mathcal{O}_{x_K} \otimes_{\mathcal{O}_{X_K}} \mathcal{L} \longrightarrow 0$$

where the last term is isomorphic to $i_*i^*\mathcal{L}$. Now $H^0(X_K, i_*i^*\mathcal{L}) = H^0(x_K, i^*\mathcal{L})$ is a one-dimensional K-vector space (this is just the stalk of $\mathcal{L}$ at x_K). Note that s not vanishing in x_K means that $i^*s \neq 0$; thus the morphism $H^0(X_K, \mathcal{L}) \to H^0(X_K, i_*i^*\mathcal{L})$ of $\Gamma(X_K, \mathcal{O}_{X_K}) = K$-modules is surjective.
So $\dim_K H^0(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = t - 1$. To conclude by induction, note that the surjectivity also implies $H^1(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = 0$ since $H^1(X_K, \mathcal{L}) = 0$ by assumption and thus $H^i(X_K, \mathcal{I} \otimes_{\mathcal{O}_{X_K}} \mathcal{L}) = 0$ for all $i > 0$ (for $i > 1$ they vanish anyway since X_K is a projective curve). Finally, note that $\mathcal{I}$ is the pullback of the ideal sheaf of the k-rational point x to X_K and all points on a smooth curve are Cartier-Divisors, so we still tensor with the pullback of a line bundle on X. □

Corollary 7.5. *The Picard functor $\mathrm{Pic}_{X/k}$ is representable by a group scheme $\underline{\mathrm{Pic}}_{X/k}$.*

7.2 The Abel-Jacobi map and the geometry of the Picard scheme

Recall the Abel-Jacobi map AJ' from Talk 1 that assignes to each effective Cartier divisor D of degree d on X a line bundle of degree d on X by sending it to $\mathcal{O}(D)$. (Recall that the degree of a line bundle $\mathcal{L}$ on C is defined as $\deg \mathcal{L} := \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$.) We want to construct a scheme version of this; we already have the scheme $\underline{\mathrm{Hilb}}^d_{X/k}$ whose k-rational points are Cartier-Divisors of degree d; now it would be nice if we had a subscheme of $\underline{\mathrm{Pic}}^d_{X/k}$ whose k-rational points are line bundles of degree d.

Definition 7.6. We define $\mathrm{Pic}^d_{X/k,\sigma}(T)$ as the subset of $\mathrm{Pic}_{X/k,\sigma}(T)$ consisting of line bundles $\mathcal{L}$ such that $\deg(X_t, \mathcal{L}_t) = d$ for all $t \in T$, where we write X_t for $X_{\kappa(t)}$ and $\mathcal{L}_t$ for $\mathcal{L}|_{X_t}$.

We claim that this is an open subfunctor. We need to check:

1. If $f : T' \to T$ is a morphism of schemes and $\mathcal{L} \in \mathrm{Pic}^d_{X/k,\sigma}(T)$, then $f_X^*\mathcal{L} \in \mathrm{Pic}^d_{X/k,\sigma}(T')$.
2. For every line bundle $\mathcal{L} \in \mathrm{Pic}_{X/k,\sigma}(T)$ there exists an open subscheme $U_{\mathcal{L},d}$ of T s.t. a morphism $f : T' \to T$ factors through $U_{\mathcal{L},d}$ if and only if $f_X^*\mathcal{L} \in \mathrm{Pic}^d_{X/k,\sigma}(T')$.

This follows from the following two results:

Lemma 7.7 (Special case of Stacks [0B59])**.** *Let K/k be an field extension. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{L}$ be a line bundle on X. Then the degree of $\mathcal{L}/X/k$ is equal to the degree of $\mathcal{L}_K/X_K/K$.*

This implies 1.. We will also use this fact extensively in the rest of this talk.
2. is then simply a consequence of

Theorem 7.8 (Special case of Stacks [0B9T])**.** *Let $f : Y \to T$ be a flat, proper morphism of finite presentation such that all fibers of f are curves and let $\mathcal{L}$ be a line bundle on Y. Then the function*

$$t \mapsto \deg \mathcal{L}_t$$

is locally constant on T.

(choose $U_{\mathcal{L},d}$ as the preimage of d under this locally constant function on T).
Note that $\coprod_{d\in\mathbb{Z}} U_{\mathcal{L},d} \cong T$.

Now observe that if $\mathcal{L}_{\text{univ}}$ is the universal element of $\mathrm{Pic}_{X/k,\sigma}$, then $U_{\mathcal{L}_{\text{univ}},d}$ represents $\mathrm{Pic}^d_{X/k,\sigma}$ (just by Yoneda formalism). Thus, we get a decomposition

$$\underline{\mathrm{Pic}}_{X/k,\sigma} = \coprod_{d\in\mathbb{Z}} \underline{\mathrm{Pic}}^d_{X/k,\sigma}$$

Also note that $\underline{\mathrm{Pic}}^d_{X/k,\sigma} \cong \underline{\mathrm{Pic}}^e_{X/k,\sigma}$ by translation with k-rational points of $\underline{\mathrm{Pic}}^{e-d}_{X/k,\sigma}$ (i.e. tensoring with line bundles $\mathcal{L}$ on X of degree $e-d$), since $\deg : \mathrm{Pic}(X) \to (\mathbb{Z},+)$ is a group homomorphism:

Lemma 7.9. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ line bundles on X s.t.* $\deg(\mathcal{L}_{1,t}) = d$ *and* $\deg(\mathcal{L}_{2,t}) = e$. *Then* $\deg(\mathcal{L}_1 \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_2) = d+e$.

Proof. Line bundles on X correspond to divisors on X: Pick an ample line bundle $\mathcal{L}_0$ on X. It has non-zero global sections, so it has a regular section, and thus there is Cartier-Divisor D_0 on X with $\mathcal{O}(D_0) = \mathcal{L}_0$. We know that there is an $n \in \mathbb{N}$ s.t. $\mathcal{L}_1 \otimes \mathcal{L}_0^{\otimes n}$ has global sections and again we get a corresponding Cartier Divisor D_1. Then $\mathcal{O}(D_1 - nD_0) \cong \mathcal{L}_1$ by our results in Talk 1.
So by weak Riemann-Roch (Talk 1), $\mathcal{L}_1$ and $\mathcal{L}_2$ correspond to divisors D_1 of degree d and D_2 of degree e. Then $D_1 + D_2$ has degree $d+e$ and corresponds to the line bundle $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$. Thus, $\deg(\mathcal{L}_1 \otimes_{O_X} \mathcal{L}_2) = d+e$ again by weak Riemann-Roch. □

Remark 7.10. Let $\mathcal{L}$ be a representative of some element in $\mathrm{Pic}_{X/k}(T)$. Then under the isomorphism $\mathrm{Pic}_{X/k}(T) \cong \mathrm{Pic}_{X/k,\sigma}(T)$, $\mathcal{L}$ maps to $\mathcal{L} \otimes_{\mathcal{O}_{X_T}} (\sigma_T \circ p_T)^*\mathcal{L}^{\otimes -1} =: \mathcal{L}^\sigma$. By the following two lemma, $\deg \mathcal{L}_t = \deg \mathcal{L}_t^\sigma$ for all t. Thus, we also get an open subfunctor $\mathrm{Pic}^d_{X/k}(T)$ of $\mathrm{Pic}_{X/k}(T)$ for all d, whose values on T consist of all equivalence classes of line bundles which have elements with fiber-wise degree d and which is isomorphic to $\mathrm{Pic}^d_{X/k,\sigma}(T)$ under the restriction of the isomorphism $\mathrm{Pic}_{X/k}(T) \cong \mathrm{Pic}_{X/k,\sigma}(T)$.

Lemma 7.11. *Let $t \in T$, $\mathcal{L}_0$ a line bundle on T and $i_X : X_{\kappa(t)} \to X_T$. Then* $\deg(X_{\kappa(t)}, i_X^* p_T^* \mathcal{L}_0) = 0$.

Proof. This is obvious by commutativity of the following diagram:

$$\begin{array}{ccc} X_t & \xrightarrow{i_X} & X_T \\ {\scriptstyle p_{\kappa(t)}}\downarrow & & \downarrow{\scriptstyle p_T} \\ \kappa(t) & \xrightarrow[i]{} & T \end{array}$$

□

Now we are ready to construct the scheme version of the Abel-Jacobi-Map.

Notation 7.12. Let $\underline{F}$ be a scheme representing the functor $F : \mathrm{Sch}^{opp} \to \mathrm{Set}$ and $Z \in F(X)$. Then we denote by $[Z]_F$ the morphism $X \to \underline{F}$ corresponding to Z. (We omit the subscript if it is clear which functor is meant.)

Remark 7.13. We will use the following fact: If we have a commutative diagram

$$\begin{array}{ccc} & & X_2 \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle [Z_2]} \\ X_1 & \xrightarrow[{[Z_1]}]{} & \underline{F} \end{array}$$

then $Z_1 = F(\varphi)(Z_2)$. This is just naturality of the isomorphism $h_{\underline{F}} \cong F$.

Fix some $d \in \mathbb{N}_0$. The identity $\underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Hilb}}^d_{X/k}$ corresponds to a closed subscheme $D_{\mathrm{univ}} \subseteq \underline{\mathrm{Hilb}}^d_{X/k} \times X$ such that the morphism $D_{\mathrm{univ}} \to \underline{\mathrm{Hilb}}^d_{X/k}$ is finite locally free of rank d.

Proposition 7.14. *Let T be a scheme over k. If $D \in \mathrm{Hilb}^d_{X/k}(T)$, then $[D]^*_X \mathcal{O}(D_{univ}) \cong \mathcal{O}(D)$.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} & & \underline{\mathrm{Hilb}}^d_{X/k} \\ & \overset{[D]}{\nearrow} & \downarrow{\scriptstyle [D_{\mathrm{univ}}]} \\ T & \xrightarrow[{[D]}]{} & \underline{\mathrm{Hilb}}^d_{X/k} \end{array}$$

By definition of the $\mathrm{Hilb}^d_{X/k}$ functor, this implies $D = D_{\mathrm{univ}} \times_{[D]_X} X_T$, i.e. the diagram

$$\begin{array}{ccc} D & \longrightarrow & D_{\mathrm{univ}} \\ \downarrow & & \downarrow \\ X_T & \xrightarrow{[D]_X} & \underline{\mathrm{Hilb}}^d_{X/k} \times X \end{array}$$

is cartesian, so with $i := [D]_X$ we have $D = i^{-1}(D_{\mathrm{univ}})$.
It remains to show that $i^*\mathcal{O}(D_{\mathrm{univ}}) = \mathcal{O}(i^{-1}D_{\mathrm{univ}})$. This holds by the following lemma. □

Lemma 7.15. *If $f : Y \to X$ is a morphism of schemes and D is a Cartier divisor on X such that $f^{-1}X$ is again a Cartier divisor, then $f^*\mathcal{O}(D) = \mathcal{O}(f^{-1}D)$.*

Proof (Sketch). Assume $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and write $D = A/(i)$ where i is a nonzero divisor. Then $f^{-1}D = B \otimes_A A/(i) \cong B/iB$. Thus, as $f^{-1}D$ is a Cartier divisor, i is still a nonzero divisor in B, which is equivalent to $(i) \otimes_A B \cong iB$. In the general case, these local isomorphisms glue to an isomorphism $f^*\mathcal{O}(-D) \cong \mathcal{O}(-f^{-1}D)$. □

Let K/k be a field extension. Then a morphism $K \to \underline{\mathrm{Hilb}}^d$ defines a Cartier divisor $D \subseteq X_K$ of degree d by Talk 4. Let $\bar{K}$ be the algebraic closure of K. $D_{X_{\bar{K}}}$ is still a Cartier divisor of degree d. By Proposition 7.14, $\mathcal{O}(D_{X_{\bar{K}}}) \cong \mathcal{O}(D_{\mathrm{univ}})|_{X_{\bar{K}}}$; and thus by (weak) Riemann-Roch (Talk 1), $\deg(X_K, \mathcal{O}(D_{\mathrm{univ}})|_{X_K}) = \deg(X_{\bar{K}}, \mathcal{O}(D_{\mathrm{univ}})|_{X_{\bar{K}}}) = d$.
So $\mathcal{O}(D_{\mathrm{univ}})$ defines an element of $\mathrm{Pic}^d_{X/k}(\underline{\mathrm{Hilb}}^d_{X/k})$ and we set $\gamma_d := [\mathcal{O}(D_{\mathrm{univ}})]$.

Corollary 7.16 (γ_d is a scheme version of AJ' from Talk 1)**.** *Let T be a scheme over k. If $D \in \mathrm{Hilb}^d_{X/k}(T)$, then $\gamma_d \circ [D] = [\mathcal{O}(D)]$.*

Proof. By definition of the Picard functor, $\gamma_d \circ [D]$ corresponds to the line bundle $[D]^*_X \mathcal{O}(D_{\mathrm{univ}})$ which is isomorphic to $\mathcal{O}(D)$ by Proposition 7.14. □

In particular, on k-rational points this is really AJ'.

Example 7.17. Since $\underline{\mathrm{Hilb}}^0_{X/k} \cong k$, γ_0 corresponds to a k-rational point of $\mathrm{Pic}^0_{X/k}$; as the unique map $k \to \underline{\mathrm{Hilb}}^0_{X/k}$ corresponds to the Cartier divisor $\emptyset$, it follows that this k-rational point corresponds to the structure sheaf on X.

Fibers of γ_d have an interesting property:

Theorem 7.18. *The fiber of γ_d over a point $p = [\mathcal{L}]$ is isomorphic to $\mathbb{P}^{\dim_{\kappa(p)} H^0(X_{\kappa(p)}, \mathcal{L})-1}_{\kappa(p)}$ if $H^0(X_{\kappa(p)}, \mathcal{L}) \neq 0$. In particular, γ_d is surjective for $d \geq g$.*

Proof (Sketch). Let $p \in \underline{\mathrm{Pic}}^d_{X/k}$, $K := \kappa(p)$ and let $\mathcal{L}$ be the line bundle on X_K of degree d corresponding to p. Consider the fiber $Z := K \times_{[\mathcal{L}]} \underline{\mathrm{Hilb}}^d_{X/k}$. Let $T = \operatorname{Spec} R$ an affine scheme, then the T-valued points of Z correspond to Cartier-Divisors $D \subseteq X_T$ with $\mathcal{O}(D) \cong \mathcal{L}|_{X_T}$ by Corollary 7.16:

$$\begin{array}{ccccc} T & \longrightarrow & Z & \longrightarrow & K \\ & \overset{[D]}{\searrow} & \downarrow & & \downarrow [\mathcal{L}] \\ & & \underline{\mathrm{Hilb}}^d_{X/k} & \xrightarrow[\gamma_d]{} & \underline{\mathrm{Pic}}^d_{X/k} \end{array}$$

First, we only consider the case $T = \operatorname{Spec} L$, where L is another field. Automorphisms of $\mathcal{O}(D)$ are precisely given by multiplication with an invertible scalar, i.e. the elements of $H^0(X_T, \mathcal{O}_{X_T})^\times \cong H^0(T, \mathcal{O}_T)^\times = L^\times$. On the other hand, the Cartier divisors D with $\mathcal{O}(D) \cong \mathcal{L}|_{X_T}$ are in 1-1-correspondenze with the regular sections of $H^0(X_T, \mathcal{L}|_{X_T}) \cong H^0(T, \mathcal{O}_T) \otimes_K H^0(X_K, \mathcal{L}) \cong L \otimes_K H^0(X_K, \mathcal{L})$ (these isomorphisms hold by flat base change), which are all non-zero sections, since X_L is still integral. This shows that the $\operatorname{Spec} L$-valued points of the fiber are in bijection with $L^{\dim H^0(X_K,\mathcal{L})} \setminus \{0\} / L^\times$, which are the $\operatorname{Spec} L$-valued points of $\mathbb{P}^{\dim_K H^0(X_K,\mathcal{L})-1}_K$.
We need to show the corresponding statement for general $T = \operatorname{Spec} R$ and naturality of this isomorphism as well to conclude the proof. For $d > 2g-2$, we can use the following argument:
There is a complex $\varphi : K_0 \to K_1$ of K-vector spaces (mentioned in Talk 6) with the following property: $H^i(X_T, \mathcal{L}|_{X_T}) = H^i(R \otimes_K K_0 \to R \otimes_K K_1)$ for all $i \in \mathbb{N}_0$ and $T = \operatorname{Spec} R$. For $d > 2g-2$, we know that $H^1(\kappa(t) \otimes_R K_0 \to \kappa(t) \otimes_R K_1) = H^1(X_{\kappa(t)}, \mathcal{L}_{\kappa(t)}) = 0$ for all $t \in T$ by Serre duality; so $\kappa(t) \otimes_R K_0 \to \kappa(t) \otimes_R K_1$ is surjective for all t, but then already $K_0 \to K_1$ is surjective by Nakayama. Thus, the complex is isomorphic to a projection $\ker \varphi \oplus K_1 \to K_1$ and we see that $E := \ker(\varphi) = H^0(X_K, \mathcal{L})$ is a finite K-vector space with $H^0(X_T, \mathcal{L}|_{X_T}) = R \otimes_K E$.
Now again, a relative Cartier divisor D on X_T with $\mathcal{O}(D) \cong \mathcal{L}|_{X_T}$ corresponds to a regular section of $\mathcal{L}|_{X_T}$, i.e. a map $s : R \to R \otimes_K E$, up to elements in $R^\times$. D is a relative effective Cartier divisor if in addition the map $D \to T$ is flat; this is the case if and only if s has a splitting, meaning that the dual map $s^v : R \otimes_K E^v \to R$ is surjective. Now we know that these surjections correspond to the T-valued points of $\mathbb{P}^{\dim H^0(X_K,\mathcal{L})-1}$ (here

it must be taken into account that $\mathcal{L}|_{X_T}$ is only determined up to line bundles on T).
We omit the proof for the case $d \leq 2g-2$. However, to conclude surjectivity for $d \geq g$, the part we proved suffices: The assumption $d \geq g$ means that $\chi(X_K, \mathcal{L}) - \chi(X_K, \mathcal{O}_{X_K}) = d \geq g = 1 - \chi(X_K, \mathcal{O}_{X_K})$, i.e. $\dim H^0(X_K, \mathcal{L}) \geq 1 + \dim H^1(X_K, \mathcal{L}) > 0$ and we showed that in this case the fiber is non-empty. ☐

Next, we want to think about the geometry of $\underline{\mathrm{Pic}}_{X/k}$. Recall that we proved that our open subfunctor F of $\mathrm{Pic}_{X/d}$ is representable by a non-empty open U of $\mathrm{Hilb}^g_{X/k}$. Actually, the proof shows that the restriction of γ_g to U is an open immersion. Moreover, we proved above that translates of U (as an open in $\underline{\mathrm{Pic}}_{X/k}$) by k-rational points of $\underline{\mathrm{Pic}}_{X/k}$ cover the whole scheme. We know that $\underline{\mathrm{Hilb}}^g_{X/k}$ is smooth of dimension g; thus, the same holds for $\underline{\mathrm{Pic}}_{X/k}$.

Corollary 7.19. *γ_d is smooth for $d > 2g-2$*

Proof. For $\deg \mathcal{L} > 2g-2$, we have $\dim_k H^0(X, \mathcal{L}) = d + 1 - g$ by Serre duality (Stacks [0B90]). Thus, all fibres of γ_d have dimension $d + 1 - g - 1 = d - g = \dim(\underline{\mathrm{Hilb}}^d_{X/k}) - \dim(\underline{\mathrm{Pic}}^d_{X/k})$, which implies that γ_d is flat (Stacks [00R4]). As all fibers are smooth, this already implies that γ_d is smooth. ☐

Next, we will prove that $\underline{\mathrm{Pic}}^d_{X/k}$ is proper over k for all $d \in \mathbb{Z}$.

Lemma 7.20. *A group scheme G over a field k is separated*

Proof. Look at the following cartesian diagram:

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{G/k}} & G \times_k G \\ \downarrow & & \downarrow \phi \\ k & \xrightarrow{\quad e \quad} & G \end{array}$$

where ϕ is the map $(g, g') \mapsto m(i(g), g')$. A k-rational point of G defines a closed immersion, so the diagonal is a closed immersion, i.e. G is separated. ☐

So $\underline{\mathrm{Pic}}_{X/k}$ is separated. For $d \geq g$, surjectivity of γ_d and separatedness of $\underline{\mathrm{Pic}}^d_{X/k}$ already imply properness of $\underline{\mathrm{Pic}}^d_{X/k}$: Indeed, as $\underline{\mathrm{Pic}}^d_{X/k}$ is smooth and thus locally of finite type, we only have to prove that $\underline{\mathrm{Pic}}^d_{X/k} \to k$ is universally closed (this implies quasi-compactness; Stacks [04XU]). But this just follows from universally closedness of $\underline{\mathrm{Hilb}}_{X/k} \to k$, as surjectivity is stable under base change.
Now properness in the case $d \geq g$ already proves properness for all d by translation (by Lemma 7.9 we just need to tensor with an invertible $\mathcal{O}_X$-module of sufficiently low degree).
Finally, note that $\underline{\mathrm{Pic}}^d_{X/k}$ is irreducible for $d \geq g$ since $\underline{\mathrm{Hilb}}^d_{X/k}$ is irreducible and γ_d is surjective, and reduced because it is smooth. Thus, it is integral. Again, by translation this holds for all d.

Corollary 7.21. *For all $d \in \mathbb{Z}$, $\underline{\mathrm{Pic}}^d_{X/k}$ is a smooth, proper, g-dimensional variety. (variety = integral + separated, finite type over k).*

By Lemma 7.9 we see that

Corollary 7.22. $\mathrm{Pic}^0_{X/k}$ *is an open and closed subgroup scheme*

Definition 7.23. Let k be a field. An abelian variety is a group scheme over k which is also a proper, geometrically integral variety over k.

Corollary 7.24. $\mathrm{Pic}^0_{X/k}$ *is an abelian variety.*

7.3 The Picard scheme of curves of genus 0 and 1

Let X be a smooth projective curve of genus 0 over an algebraically closed field k. Then γ_0 is an isomorphism, because γ_0 is surjective and the nonempty open of $\underline{\mathrm{Hilb}}^0_{X/k}$ on which γ_0 induces an open immersion has to be $\underline{\mathrm{Hilb}}^0_{X/k} = k$ itself. Thus, for all $d \in \mathbb{Z}$, $\underline{\mathrm{Pic}}^d_{X/k} \cong \underline{\mathrm{Pic}}^0_{X/k} \cong \operatorname{Spec} k$. In particular, for each $d \in \mathbb{Z}$ and every scheme T over k there is exactly one line bundle of degree d on $\mathcal{O}_{X_T}$; thus, $\mathrm{Pic}(X_T) = \mathbb{Z}$. One can show that the only smooth projective curve of genus 0 over an algebraically closed field is $\mathbb{P}^1_k$.
More interesting is the case $g = 1$.

Definition 7.25. An elliptic curve is a smooth projective curve of genus 1 with a distinguished point

We know that γ_1 is surjective. Note that $\underline{\mathrm{Hilb}}^1_{X/k} \cong X$ and $\underline{\mathrm{Pic}}^1_{X/k}$ is a smooth, proper, one-dimensional variety, so γ_1 is a dominant map of proper normal curves. Now recall that then $\underline{\mathrm{Hilb}}^1_{X/k} \cong \underline{\mathrm{Pic}}^1_{X/k}$ (via γ_1) if γ_1 induces an isomorphism of function fields (we have an equivalence of categories {Proper, normal curves over k with dominant morphisms} $\leftrightarrow$ {Fin. generated field extensions of k with transcendence degree 1}). But as there is some non-empty open of $\underline{\mathrm{Hilb}}^1_{X/k}$ on which γ_1 is an open immersion, this holds; so $\underline{\mathrm{Pic}}^d_{X/k} \cong \underline{\mathrm{Pic}}^1_{X/k} \cong X$ for all $d \in \mathbb{Z}$.
The beautiful thing about this is that we know that $\underline{\mathrm{Pic}}^0_{X/k}$ is an abelian variety; so X is naturally equipped with a group structure.